# The Analysis of a Broad-Band Circular Polarizer Including Interface Reflections*

S. ADACHI†, MEMBER, IRE, AND E. M. KENNAUGH, ASSOCIATE MEMBER, IRE†

*Summary*—**Transmission characteristics of a broad-band circular polarizer consisting of anisotropic dielectric plates are rigorously analyzed. The transmitted wave is formulated in terms of the incident wave including interface reflections. The influence of the interface reflections on the axial ratio of the polarization is numerically shown vs frequency. Frequency dependence of the power transmission ratio is also obtained. From the above analysis, it can be concluded that a circular polarizer of this type is promising as a new broad-band circular polarizer.**

## Introduction

It was shown by Pancharatnam[1] that a combination of birefringent plates transforms linear polarization to circular polarization over a very broad optical frequency band. The possibility of applying this principle to a circular polarizer in the microwave frequency range was discussed in detail by Long.[2]

This circular polarizer consists of three cascaded half- and/or quarter-wave birefringent plates, or anisotropic dielectric plates with their axes oriented at definite angles with respect to the incident polarization. Complete circular polarization can be obtained at three different points of frequency by using three plates, each having a different tilt angle with respect to the incident plane wave. Generally, it will be possible to design an $n$-plate circular polarizer having an axial ratio of 1 at $n$ different frequencies.

Analyses by Pancharatnam and Long are confined to the ideal cases where the interface reflections are neglected. The reflections at these interfaces are inevitable in practical use, particularly for a polarizer consisting of relatively thin plates with high dielectric constant. It is easily conceivable that even small reflections have significant effects on the axial ratio of the emerging wave.

In this report, the transmission characteristics of the circular polarizer, including interface reflections, are rigorously analyzed. The influence of the interface reflections on axial ratio vs frequency is numerically shown. Frequency dependence of the power transmission ratio is also obtained.

* Received by the PGMTT, March 21, 1960; revised manuscript received, June 13, 1960.

† Antenna Lab., Dept. of Elec. Engrg., The Ohio State University, Columbus.

[1] S. Pancharatnam, "Achromatic combination of birefringent plates," *Proc. Indian Acad. Sci.*, Sec. A, vol. 41, p. 130 ff.; April, 1955.

[2] R. K. Long, "Some methods of obtaining circular polarization over a broad band of frequencies by the use of birefringent plates," Antenna Lab., The Ohio State Univ. Res. Foundation, Contract AF 33(616)-5078, Wright Air Dev. Ctr., Wright-Patterson A.F.B., Dayton, Ohio. (AD 203827), Rept. No. 768-2; June 1, 1958.

## Birefringent-Plate Circular Polarizer

The mechanism of the birefringent-plate circular polarizer can be well described by a geometrical representation on the Poincaré Sphere.[3,4] An elliptical polarization with axial ratio $r$ ($\geq 1$) and tilt angle $\beta$ ($0° \leq \beta \leq 180°$) between the major axis and a reference axis can be uniquely represented by a point having $2\beta$ and $2\alpha = \pm 2 \cot^{-1} r$ as longitude and latitude, respectively, on a sphere. A positive sign is chosen for $\alpha$ when the polarization is of left-hand sense; a negative sign is chosen for $\alpha$ when the polarization is of right-hand sense. Such a sphere is called the Poincaré Sphere, and is shown in Fig. 1. The equator represents linear polarizations. All polarizations of left-hand sense lie in the upper hemisphere ($0° \leq 2\alpha \leq 90°$) and all polarizations of right-hand sense lie in the lower hemisphere ($-90° \leq 2\alpha \leq 0°$). A change from one state of polarization to another is represented by a rotation of the sphere about an appropriate diameter, so long as the transmission is lossless. For instance, passage of any polarization through a half-wave birefringent plate corresponds to a rotation of 180° about an axis in the equatorial plane. Similarly, passage through a quarter-wave plate corresponds to rotation of 90° about an axis in the equatorial plane. In particular, the passage of a linearly polarized wave through a quarter-wave plate whose axes are at 45° to the direction of polarization results in circular polarization, represented by the pole of the sphere. The transformation of a linearly polarized wave into a circularly polarized wave by the three-birefringent plate-

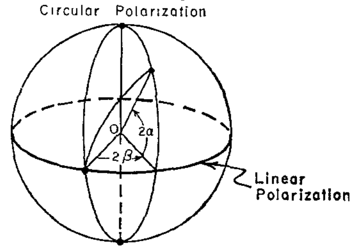


Fig. 1—Poincaré Sphere.

[3] H. Poincaré, "Theorie Mathematique de la Lumière," g. Carré, Paris; 1889.

[4] G. A. Deschamps, "Techniques for handling elliptically polarized waves with special reference to antennas," Proc. IRE, vol. 39, pp. 540–544; May, 1951.

circular polarizer is illustrated by the Poincaré Sphere in Fig. 2. At the central wavelength $\lambda_0$, the incident linearly polarized wave represented by a point $P$ is transformed to a point $Q$ by passage through half-wave plate No. 1 (rotation of 180° about axis $00_1$), and next, to a point $R$ by passage through half-wave plate No. 2 (rotation of 180° about axis $00_2$). Finally, the linear polarization $R$ is transformed to circular polarization at the point $S$ by passage through a quarter-wave plate (rotation of 90° about axis $00_3$). The axes $00_1$, $00_2$, and $00_3$ are determined by the orientation of the fast axes of the individual plates with respect to the reference direction. At the equideviated wavelengths $\lambda_1$ and $\lambda_2$ from the central wavelength, the point $P$ is transformed through points $Q_1$, $R_1$, (rotations of $180°-2\delta$ about $00_1$, $00_2$), or $Q_2$, $R_2$ (rotations of $180°+2\delta$ about $00_1$, $00_2$), respectively, to the same point $S$ (rotations of $90°-\delta$ or $90°+\delta$ about $00_3$). Thus, it is possible to design a three-plate circular polarizer such that a fixed direction of linear polarization is transformed to complete circular polarization at three different frequencies. However, this analysis (originally given by Pancharatnam) does not include the effect of interfacial reflections.

## Rigorous Analysis of a Broad-Band Circular Polarizer

The plates are assumed to be infinite, and the direction of propagation of the incident wave is assumed to be normal to the plate. In Fig. 3, the plate is tilted by an angle $\phi$ with respect to the coordinate $x$. Coordinates $x'$ and $y'$ are the fast and slow axes of the plate, respectively. The "fast" axis is referred to the polarization direction of the wave which has the highest phase velocity, or the direction along which the dielectric constant is minimum. A right-hand coordinate system is used.

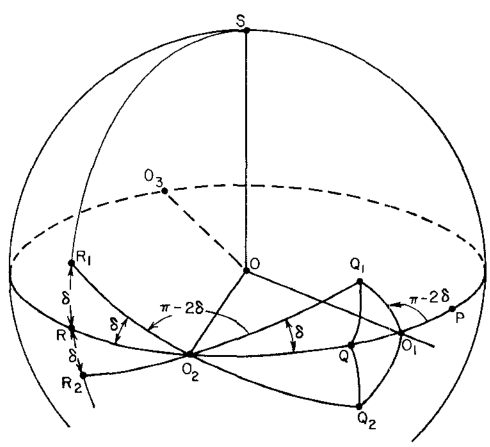


Fig. 2—Map of three-plate circular polarizer on the Poincaré sphere.

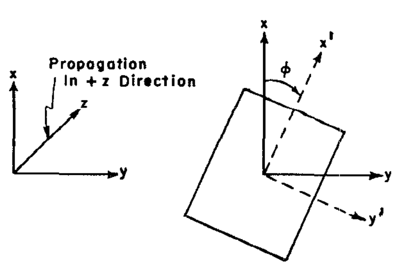


Fig. 3—Coordinate system.

Transmission matrices $A_{1f}$ and $A_{1s}$, associated with fast and slow axes of the plate 1 respectively, are defined as follows:

$$A_{1f} = \begin{bmatrix} \cos k_{1f}d_1 & jZ_{1f} \sin k_{1f}d_1 \\ \frac{j}{Z_{1f}} \sin k_{1f}d_1 & \cos k_{1f}d_1 \end{bmatrix}, \quad (1)$$

and

$$A_{1s} = \begin{bmatrix} \cos k_{1s}d_1 & jZ_{1s} \sin k_{1s}d_1 \\ \frac{j}{Z_{1s}} \sin k_{1s}d_1 & \cos k_{1s}d_1 \end{bmatrix}, \quad (2)$$

where $k_{1f}$, $k_{1s}$ are phase constants associated with fast and slow axes respectively, and $Z_{1f}$, $Z_{1s}$ are normalized characteristic impedances of the respective media with reference to the free-space characteristic impedance. The thickness of plate 1 is given by $d_1$.

Since linearly polarized plane waves along each of the perpendicular axes $x'$ and $y'$ travel independently, we obtain the following relation using the transmission matrices defined above:

$$\begin{bmatrix} E_x' \\ H_y' \\ E_y' \\ -H_x' \end{bmatrix}_{z=0} = T_{10} \begin{bmatrix} E_x' \\ H_y' \\ E_y' \\ -H_x' \end{bmatrix}_{z=d_1}, \quad (3)$$

where

$$T_{10} = \begin{pmatrix} A_{1f} & 0 \\ 0 & A_{1s} \end{pmatrix}, \quad (4)$$

and where $\mathbf{0}$ is a zero matrix of 2 rows and 2 columns. The coordinate transformation matrix $\phi_1$ from coordinates $(x', y', z')$ into coordinates $(x, y, z)$ is expressed as follows:

$$\phi_1 = \begin{pmatrix} \cos \phi_1 I & -\sin \phi_1 I \\ \sin \phi_1 I & \cos \phi_1 I \end{pmatrix}, \quad (5)$$

where $I$ is a unit matrix of 2 rows and 2 columns. By using the above matrix, the following transmission equation for the field components referred to the unprimed coordinate system is obtained:

$$\begin{bmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{bmatrix}_{z=0} = W_{10} \begin{bmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{bmatrix}_{z=d_1}, \quad (6)$$

where

$$W_{10} = \phi_1 T_{10} \phi_1^{-1}, \tag{7}$$

and where $\phi_1^{-1}$ is the inverse matrix of $\phi_1$. $W_{10}$ is the transmission matrix associated with the plate 1 whose fast axis is tilted by angle $\phi_1$ from the $x$ axis toward the $y$ axis.

Carrying out the matrix calculation of (7), the following expression results for $W_{10}$:

$$2W_{10} = \cos 2\phi_1 \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix} + \sin 2\phi_1 \begin{pmatrix} 0 & A_1 \\ A_1 & 0 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix}, \tag{8}$$

where

$$A_1 = A_{1f} - A_{1s},$$
$$B_1 = A_{1f} + A_{1s}. \tag{9}$$

The transmission through a circular polarizer consisting of $n$ plates is represented by the $n$ products of matrices of the above type; that is,

$$\begin{bmatrix} E_{ox} \\ H_{oy} \\ E_{oy} \\ -H_{ox} \end{bmatrix} = W_{01}W_{12} \cdots W_{n-1n} \begin{bmatrix} E_{nx} \\ H_{ny} \\ E_{ny} \\ -H_{nx} \end{bmatrix}, \tag{10}$$

where subscripts $_o$, $_n$ are referred to the fields at sending and receiving end-surfaces respectively as illustrated in Fig. 4(a).

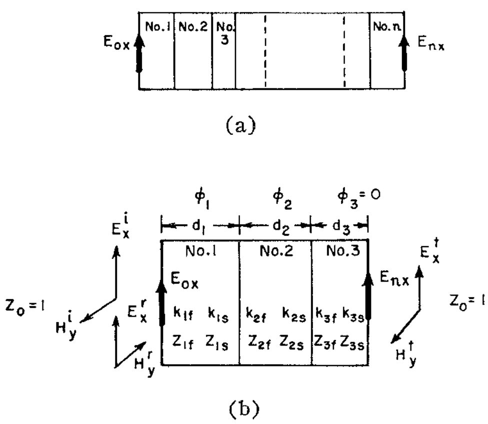


Fig. 4—(a) An $n$-plate circular polarizer, (b) A three-plate circular polarizer.

Now let $E_{ox}$, $E_{nx}$, etc., be expressed in terms of the incident, reflected, and transmitted fields as follows:

$$E_{ox} = E_x^i + E_x^r,$$
$$E_{oy} = E_y^i + E_y^r,$$
$$H_{oy} = H_y^i - H_y^r = E_x^i - E_x^r,$$
$$-H_{ox} = H_x^i - \mathrm{H}_x^r = E_y^i - E_y^r, \tag{11}$$

and

$$E_{nx} = E_x^t,$$
$$E_{ny} = E_y^t,$$
$$H_{ny} = H_y^t = E_x^t,$$
$$-H_{nx} = H_x^t = E_y^t. \tag{12}$$

It is noted here that since input and output media are assumed to be free space, their normalized characteristic impedances are equal to 1.

Substituting (11) and (12) in (10) yields the following equation for a three-plate circular polarizer:

$$\begin{bmatrix} E_x^i + E_x^r \\ E_x^i - E_x^r \\ E_y^i + E_y^r \\ E_y^i - E_y^r \end{bmatrix} = W_{01}W_{12}W_{23} \begin{bmatrix} E_x^t \\ E_x^t \\ E_y^t \\ E_y^t \end{bmatrix}. \tag{13}$$

It is possible to set $\phi_3 = 0$ without losing generality; therefore,

$$W_{23} = T_{23}.$$

Carrying out matrix multiplications yields

$$4W_{01}W_{12}W_{23} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \tag{14}$$

where

$$A = [\cos 2(\phi_1 - \phi_2)A_1A_2 + \cos 2\phi_1 A_1B_2 + \cos 2\phi_2 B_1A_2 + B_1B_2]A_{3f},$$
$$B = [\sin 2(\phi_1 - \phi_2)A_1A_2 + \sin 2\phi_1 A_1B_2 + \sin 2\phi_2 B_1A_2]A_{3f}$$
$$C = [-\sin 2(\phi_1 - \phi_2)A_1A_2 + \sin 2\phi_1 A_1B_2 + \sin 2\phi_2 B_1A_2]A_{3s},$$
$$D = [\cos 2(\phi_1 - \phi_2)A_1A_2 - \cos 2\phi_1 A_1B_2 - \cos 2\phi_2 B_1A_2 + B_1B_2]A_{3s}. \tag{15}$$

Solving (13) for the transmitted field ($E_x^t$, $E_y^t$) in terms of the incident field ($E_x^i$, $E_y^i$), yields

$$\begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix} = 8 \frac{\begin{pmatrix} \sum_{ij} D_{ij} & -\sum_{ij} C_{ij} \\ -\sum_{ij} B_{ij} & \sum_{ij} A_{ij} \end{pmatrix}}{\begin{vmatrix} \sum_{ij} A_{ij} & \sum_{ij} C_{ij} \\ \sum_{ij} B_{ij} & \sum_{ij} D_{ij} \end{vmatrix}} \begin{pmatrix} E_x^i \\ E_y^i \end{pmatrix},$$

$$i = 1, 2;\ j = 1, 2, \tag{16}$$

and for the reflected field

$$\begin{pmatrix} E_x^r \\ E_y^r \end{pmatrix} = \frac{1}{8} \begin{pmatrix} \sum_j A_{1j} - \sum_j A_{2j} & \sum_j C_{1j} - \sum_j C_{2j} \\ \sum_j B_{1j} - \sum_j B_{2j} & \sum_j D_{1j} - \sum_j D_{2j} \end{pmatrix} \begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}, \tag{17}$$

where $A_{ij}$, etc., indicate matrix elements of matrices $\boldsymbol{A}$, etc. Right- and left-circularly polarized components of the transmitted wave are given by the following form:

$$R = \frac{1}{2}(1 \ -j)\begin{pmatrix}E_x^t \\ E_y^t\end{pmatrix},$$

$$L = \frac{1}{2}(1 \quad j)\begin{pmatrix}E_x^t \\ E_y^t\end{pmatrix}. \tag{18}$$

Axial ratio of polarization of the transmitted wave, A.R., is calculated by

$$\text{A.R.} = \frac{|L| + |R|}{|L| - |R|}. \tag{19}$$

Power transmission ratio, P.T.R., is given by

$$\text{P.T.R.} = \frac{|E_x^t|^2 + |E_y^t|^2}{|E_x^i|^2 + |E_y^i|^2}. \tag{20}$$

## Some Numerical Results

Numerical calculation has been carried out for a circular polarizer consisting of two half-wave plates (No. 1 and No. 2) and one quarter-wave plate (No. 3). Tilt angles $\phi_1$, $\phi_2$ and incident field components $E_x^i$, $E_y^i$ have been chosen such that in the absence of interface reflections, perfect circular polarization is obtained at three equispaced wavelengths: $\lambda_0$ (central wavelength), $\lambda_1$ and $\lambda_2$, where

$$\lambda_1 = \lambda_0\left(1 - \frac{\delta}{90°}\right),$$

and

$$\lambda_2 = \lambda_0\left(1 + \frac{\delta}{90°}\right). \tag{21}$$

This requires:[1,2]

$$\phi_1 = \tfrac{1}{2}(90° + 2a + e),$$

$$\phi_2 = \tfrac{1}{2}(90° + a),$$

$$E_x^i = \cos\tfrac{1}{2}(90° + 2a + 2e),$$

$$E_y^i = \sin\tfrac{1}{2}(90° + 2a + 2e), \tag{22}$$

where $a$ is a solution of the equation:

$$A\sin^2 a + B\sin a + C = 0, \tag{23}$$

and where

$$A = 4\cos^3\delta(1 - \cos\delta),$$

$$B = 4\cos^2\delta(\cos 2\delta - \cos\delta),$$

$$C = 2\cot^2\delta(1 - \cos\delta) - \cos^2 2\delta, \tag{24}$$

and $e$ in (22) is given by

$$\sin e = \frac{2\cos^2\delta\sin a - \cos 2\delta}{2\cos\delta}.$$

The characteristic values of four polarizers are shown in Table I, for several choices of dielectric constant along fast and slow axes and different values of $\delta$ (which determines the bandwidth as well as the orientation angles of the plates). The composition of all plates in each example is assumed the same. The particular combination of the dielectric constants in (3) and (4) are those reported for a stratified polystyrene-styrofoam medium.[5]

TABLE I

The Characteristic Values of the Circular Polarizers

| No. of Example | (1) | (2) | (3) | (4) |
|---|---|---|---|---|
| $\epsilon_f' = \frac{\epsilon_f}{\epsilon_0}$ | 1.0 | 1.0 | 1.470 | 1.470 |
| $\epsilon_s' = \frac{\epsilon_s}{\epsilon_0}$ | 1.2 | 1.5 | 1.795 | 1.795 |
| $k_0 d_1 = \frac{\pi}{\sqrt{\epsilon_s'} - \sqrt{\epsilon_f'}}$ | 32.915 | 13.978 | 24.671 | 24.671 |
| $\delta$ | 18° | 18° | 18° | 23.5° |
| $\phi_1$ | 94.654° | 94.654° | 94.654° | 92.493° |
| $\phi_2$ | 66.575° | 66.575° | 66.575° | 65.172° |
| $\tan^{-1}\frac{E_y^i}{E_x^i}$ | 101.159° | 101.159° | 101.159° | 99.642° |

The axial ratios of the transmitted wave for the above four examples have been calculated by using the preceding equations. The calculation of the matrix products was programmed on the IBM 650 computer by H. W. Baeumler. The results are plotted in Figs. 5–8, and are compared with the axial ratios in the case of no reflection. These curves show very complex behavior for the frequency variation because of interference between multiply reflected waves. The effect of interference reflection is to increase the axial ratio of the output polarization, although in the worst case the resulting axial ratio is less than 1.075 over a 2:1 band of frequencies.

The axial ratio of polarizer No. 1 with the ratio of the dielectric constants, $\epsilon_s/\epsilon_f = 1.2$ is much better than that of polarizer No. 3 with almost the same ratio, $\epsilon_s/\epsilon_f = 1.22$. This shows that the absolute value of the dielectric con-

[5] H. S. Kirschbaum and S. Chen, "A method of producing broad-band circular polarization employing an anisotropic dielectric," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 199–203; July, 1957.

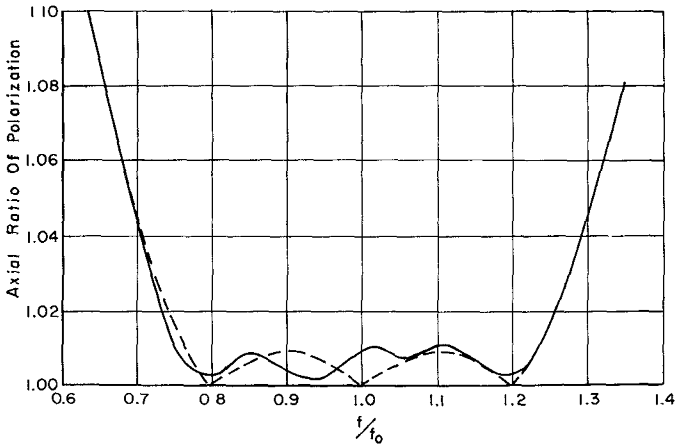


Fig. 5—Axial ratio of polarization vs frequency for Example (1). —— Interface reflection included; – – – Interface reflection neglected.

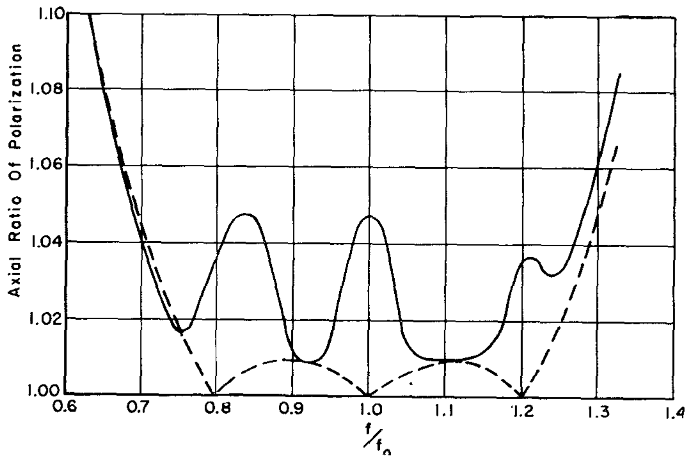


Fig. 6—Axial ratio of polarization vs frequency for Example (2). —— Interface reflection included; – – – Interface reflection neglected.

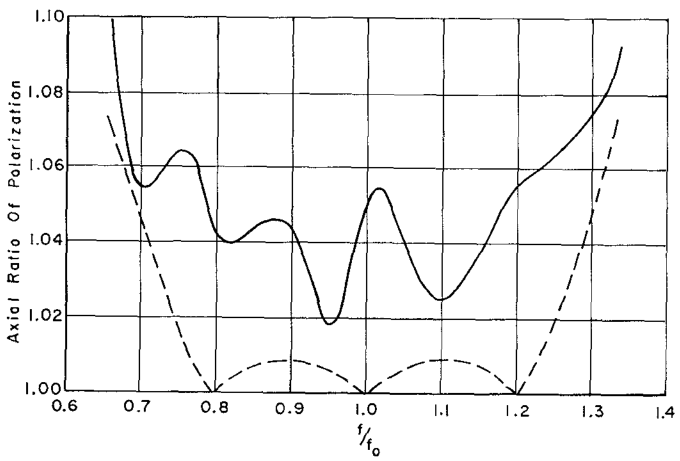


Fig. 7—Axial ratio of polarization vs frequency for Example (3). —— Interface reflection included; – – – Interface reflection neglected.

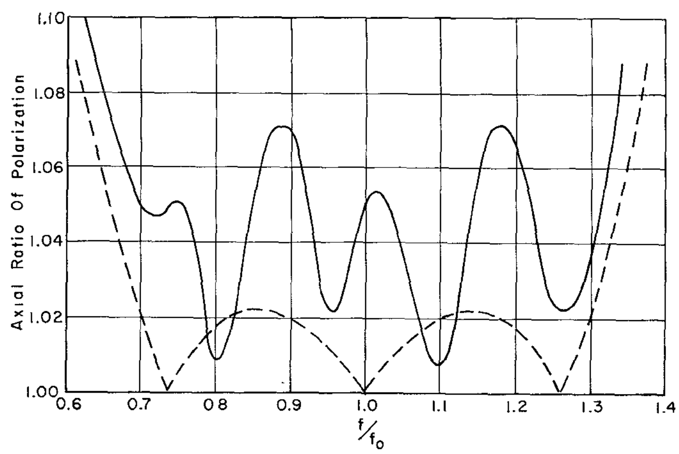


Fig. 8—Axial radio of polarization vs frequency for Example (4). —— Interface reflection included; – – – Interface reflection neglected.

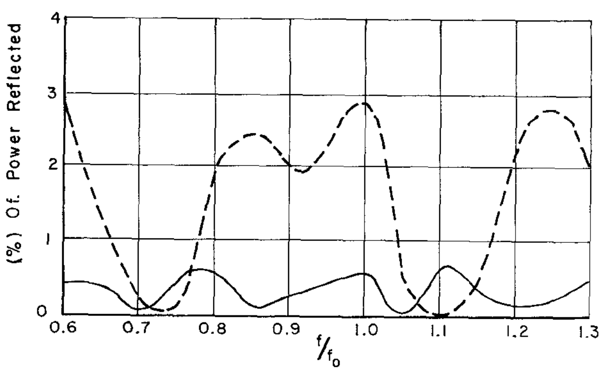


Fig. 9—Percentage of power reflected vs frequency. —— Example (1); – – – Example (2).

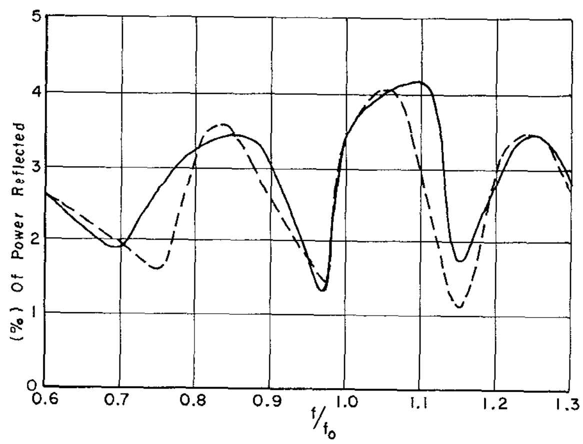


Fig. 10—Percentage of power reflected vs frequency. ——Example (3); – – – Example (4).

stant is as important as the ratio, $\epsilon_s/\epsilon_f$, and must be as small as possible to obtain complete circular polarization. With the increase of the dielectric constants ($\epsilon_s/\epsilon_f$ = constant), the reflection at the receiving end surface (boundary between the plate No. 3 and the free space) increases and causes further multiple reflections and interactions between the sending and receiving end surfaces. It is noted also that the output axial ratio of polarizer No. 1 is extremely good, considering the depolarization which would be obtained for a circular polarized wave passing through a single plane interface. The above tells us that in this case the multiple reflections at the boundaries compensate each other as a whole so as to transmit a complete circular polarized wave over a wide frequency band.

The percentages of power reflected are plotted in Figs. 9 and 10. It is seen that better than 95 per cent transmission is obtained in all cases, and that the combinations yielding poorer axial ratios also exhibit the greater reflection.

## Conclusions

A broad-band circular polarizer consisting of three anisotropic dielectric plates has been rigorously analyzed under the assumption that each plate has constant (different) values of refractive index along the two principal axes. As a result, the transmitted wave has been formulated in terms of the incident wave including interface reflections. The frequency characteristics of the axial ratio of polarization and the power transmission ratio have been numerically shown. From the above analysis it can be concluded that a combination of this type is very promising as a broad-band circular polarizer. The axial ratio is less than 1.075 and the reflected power less than 5 per cent over more than a 2:1 band of frequencies for each of the anisotropic dielectric plate combinations chosen as examples in this paper.

## Acknowledgment

The assistance of Mrs. F. Phillips in the computation is gratefully acknowledged.

# Large Signal Analysis of a Parametric Harmonic Generator*

KENNETH M. JOHNSON†

*Summary*—**Large signal analysis of a harmonic generator using a semiconductor diode reveals a larger possible efficiency than a similar small signal analysis. As higher harmonic numbers are reached, large signal analysis becomes increasingly more important in predicting the maximum conversion efficiency. It is shown that there exists an optimum value for the diode bias voltage and an optimum coupling of the load and generator to the diode, and that the diode operating voltage should almost drive the diode into conduction. An expression is derived for the maximum conversion efficiency for any harmonic, and it is shown that the conversion loss increases with increasing harmonic number, approximately 2.9 db per *n* for large harmonic numbers in a typical case.**

## Introduction

Harmonic generators, unlike parametric amplifiers, are generally large signal devices; that is, the input signal traverses the entire dynamic range of the diode. For this reason, a large signal analysis must be used in the determination of the maximum conversion efficiency, especially at high harmonic numbers. Large signal analysis takes into account all the higher order contributions of the diode nonlinearity to the efficiency, and, as a result, a greater value for the maximum efficiency is obtained than is revealed by small signal analysis. The large signal analysis is reduced to the small signal equivalent[1] when higher order terms are not considered. This paper describes the large signal analysis of a semiconductor diode operating as a frequency converter and discusses the conditions required for maximum conversion efficiency.

## Semiconductor Diode Harmonic Production

The Manley-Rowe[2] power relations indicate that a nonlinear reactance in a circuit can give rise to frequency conversion without losses. A back-biased semiconductor diode behaves as a nonlinear reactance in a circuit and, as a result, is useful for frequency conversion.[3] In this section we shall consider a diode as a harmonic generator to see how much energy conversion may be expected.

* Received by the PGMTT, April 6, 1960; revised manuscript received, June 28, 1960.

† Microwave Lab., Hughes Aircraft Co., Culver City, Calif.

[1] D. B. Leeson and S. Weinreb, "Frequency multiplication with nonlinear capacitance—a circuit anslysis," Proc. IRE, vol. 47, pp. 2076–2084; December, 1959.

[2] J. M. Manley and H. E. Rowe, "Some general properties of nonlinear elements—part I, general energy relations," Proc. IRE, vol. 44, pp. 904–913; July, 1956.

[3] D. Leenov and A. Uhlir, "Generation harmonics and subharmonics at microwave frequencies with *p-n* junction diodes," Proc. IRE, vol. 47, pp. 1724–1729; October, 1959.